Introduction to modal logic

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A logic

A language

i.e. a collection of well-formed expressions to which meaning can be assigned.

A semantics

describing how language expressions are interpreted as statements about something.

A deductive system

i.e. a collection of rules to derive in a purely syntactic way facts and relationships among semantic objects described in the language.

Note

- a purely syntactic approach (up to the 1940's; the sacred form)
- a model theoretic approach (A. Tarski legacy)

Semantic reasoning: models

- sentences
- models & satisfaction: $\mathfrak{M} \models \phi$
- validity: $\models \phi$ (ϕ is satisfied in every possible structure)
- logical consequence: $\Phi \models \phi$ (ϕ is satisfied in every model of Φ)
- theory: $Th\,\Phi$ (set of logical consequences of a set of sentences Φ)

Syntactic reasoning: deductive systems

Deductive systems $\vdash$

- sequents
- Hilbert systems
- natural deduction
- tableaux systems
- resolution
- ...

- derivation and proof
- deductive consequence: $\Phi \vdash \phi$
- theorem: $\vdash \phi$

Soundness & completeness

- A deductive system $\vdash$ is sound wrt a semantics $\models$ if for all sentences ϕ

$$\vdash \phi \implies \models \phi$$

(every theorem is valid)

- $\cdots$ complete ...

$$\models \phi \implies \vdash \phi$$

(every valid sentence is a theorem)

Consistency & refutability

For logics with negation and a conjunction operator

- A sentence ϕ is refutable if $\neg\phi$ is a theorem (i.e. $\vdash \neg\phi$)
- A set of sentences Φ is refutable if some finite conjunction of elements in Φ is refutable
- ϕ or Φ is consistent if it is not refutable.

Examples

$$\mathfrak{M} \models \phi$$

- Propositional logic (logic of uninterpreted assertions; models are truth assignments)
- Equational logic (formalises equational reasoning; models are algebras)
- First-order logic (logic of predicates and quatification over structures; models are relational structures)
- Modal logics
- ...

Modal logic (from P. Blackburn, 2007)

Over the years modal logic has been applied in many different ways. It has been used as a tool for reasoning about time, beliefs, computational systems, necessity and possibility, and much else besides.

These applications, though diverse, have something important in common: the key ideas they employ (flows of time, relations between epistemic alternatives, transitions between computational states, networks of possible worlds) can all be represented as simple graph-like structures.

Modal logics are

- tools to talk about relational, or graph-like structures.
- fragments of classical ones, with restricted forms of quantification ...
- ... which tend to be decidable and described in a pointfree notations.

The language

Syntax

$$\phi \ ::= \ p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \to \phi_2 \mid \langle m \rangle\, \phi \mid [m]\, \phi$$

where $p \in$ PROP and $m \in$ MOD

Disjunction ($\vee$) and equivalence ($\leftrightarrow$) are defined by abbreviation. The signature of the basic modal language is determined by sets PROP of propositional symbols (typically assumed to be denumerably infinite) and MOD of modality symbols.

The language

Notes

- if there is only one modality in the signature (i.e., MOD is a singleton), write simply $\Diamond\phi$ and $\Box\phi$
- the language has some redundancy: in particular modal connectives are dual (as quantifiers are in first-order logic): $[m]\,\phi$ is equivalent to $\neg\langle m\rangle\,\neg\phi$
- define modal depth in a formula ϕ, denoted by $\mathrm{md}\,\phi$ as the maximum level of nesting of modalities in ϕ

Example

Models as LTSs over Act.
$MOD = \mathbb{P}Act$ – sets of actions.
$\langle\{a,b\}\rangle\,\phi$ can be read as "*after observing a or b, ϕ must hold.*"
$[\{a,b\}]\,\phi$ can be read as "*after observing a and b, ϕ must hold.*"

Semantics

$\mathfrak{M}, w \models \phi$ – **what does it mean?**

Model definition

A model for the language is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where

- $\mathfrak{F} = \langle W, \{R_m\}_{m \in \mathrm{MOD}} \rangle$
  is a Kripke frame, ie, a non empty set W and a family R_m of binary relations (called *accessibility relations*) over W, one for each modality symbol $m \in \mathrm{MOD}$. Elements of W are called points, states, worlds or simply vertices in directed graphs.
- $V : \mathrm{PROP} \longrightarrow \mathcal{P}(W)$ is a valuation.

When MOD $= \mathbf{1}$

- $\Diamond\phi$ and $\Box\phi$ instead of $\langle\cdot\rangle\,\phi$ and $[\cdot]\,\phi$
- $\mathfrak{F} = \langle W, R \rangle$ instead of $\mathfrak{F} = \langle W, \{R_m\}_{m \in \mathrm{MOD}} \rangle$

Semantics

Safistaction: for a model $\mathfrak{M}$ and a point w

$$
\begin{array}{lll}
\mathfrak{M}, w \models \text{true} & & \\
\mathfrak{M}, w \not\models \text{false} & & \\
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \models \neg\phi & \text{iff} & \mathfrak{M}, w \not\models \phi \\
\mathfrak{M}, w \models \phi_1 \wedge \phi_2 & \text{iff} & \mathfrak{M}, w \models \phi_1 \text{ and } \mathfrak{M}, w \models \phi_2 \\
\mathfrak{M}, w \models \phi_1 \rightarrow \phi_2 & \text{iff} & \mathfrak{M}, w \not\models \phi_1 \text{ or } \mathfrak{M}, w \models \phi_2 \\
\mathfrak{M}, w \models \langle m \rangle \phi & \text{iff} & \text{there exists } v \in W \text{ st } wR_m v \text{ and } \mathfrak{M}, v \models \phi \\
\mathfrak{M}, w \models [m] \phi & \text{iff} & \text{for all } v \in W \text{ st } wR_m v \text{ and } \mathfrak{M}, v \models \phi
\end{array}
$$

Semantics

Satisfaction

A formula ϕ is

- satisfiable in a model $\mathfrak{M}$ if it is satisfied at some point of $\mathfrak{M}$
- globally satisfied in $\mathfrak{M}$ ($\mathfrak{M} \models \phi$) if it is satisfied at all points in $\mathfrak{M}$
- valid ($\models \phi$) if it is globally satisfied in all models
- a semantic consequence of a set of formulas Γ ($\Gamma \models \phi$) if for all models $\mathfrak{M}$ and all points w, if $\mathfrak{M}, w \models \Gamma$ then $\mathfrak{M}, w \models \phi$

Example: Hennessy-Milner logic

Process logic (Hennessy-Milner logic)

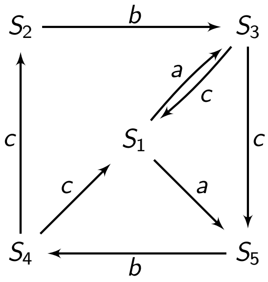
- $\mathrm{PROP} = \emptyset$
- $W = \mathbb{P}$ is a set of states, typically process terms, in a labelled transition system
- each subset $K \subseteq Act$ of actions generates a modality corresponding to transitions labelled by an element of K

Assuming the underlying LTS $\mathfrak{F} = \langle \mathbb{P}, \{p \xrightarrow{K} p' \mid K \subseteq Act\} \rangle$ as the modal frame, satisfaction is abbreviated as

$$p \models \langle K \rangle \phi \quad \text{iff} \quad \exists_{q \in \{p' \mid p \xrightarrow{a} p' \,\wedge\, a \in K\}} \,.\, q \models \phi$$

$$p \models [K] \phi \quad \text{iff} \quad \forall_{q \in \{p' \mid p \xrightarrow{a} p' \,\wedge\, a \in K\}} \,.\, q \models \phi$$

Example: Hennessy-Milner logic



Prove:

1. $S_2 \models [a]\,(\langle b\rangle\, tt \wedge \langle c\rangle\, tt)$
2. $S_1 \not\models [a]\,(\langle b\rangle\, tt \wedge \langle c\rangle\, tt)$
3. $S_2 \models [b]\,[c]\,(\langle a\rangle\, tt \vee \langle b\rangle\, tt)$
4. $S_1 \models [b]\,[c]\,(\langle a\rangle\, tt \vee \langle b\rangle\, tt)$

Examples I

An automaton

$$A \;=\; \begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[loop above, "a"] \arrow[r, "b"] & 3 \arrow[loop below, "b"] \end{tikzcd}$$

- two modalities $\langle a \rangle$ and $\langle b \rangle$ to explore the corresponding classes of transitions
- note that
$$1 \models \langle a \rangle \cdots \langle a \rangle \langle b \rangle \cdots \langle b \rangle\, t$$
where t is a proposition valid only at the (terminal) state 3.
- all modal formulas of this form correspond to the strings accepted by the automaton, i.e. in language $\mathcal{L} = \{a^m b^n \mid m, n > 0\}$

Examples II

$(P, <)$ a strict partial order with infimum 0

- $P, x \models \Box$false if x is a maximal element of P
- $P, 0 \models \Diamond \Box \mathit{false}$ iff ...
- $P, 0 \models \Box \Diamond \Box \mathit{false}$ iff ...

Examples III

Temporal logic

- $\langle T, <\rangle$ where T is a set of time points (instants, execution states , ...) and $<$ is the earlier than relation on T.
- Thus, $\Box\varphi$ (respectively, $\Diamond\varphi$) means that φ holds in all (respectively, some) time points.

Examples III

$\langle T, < \rangle$

The structure of time is a strict partial order
(i.e., a transitive and asymmetric relation)

For any such structure, a new modality, $\bigcirc$, can be defined based on the cover relation $\lessdot$ for $<$ (*i.e.*, $x \lessdot y$ if (1) every $x < y$ and (2) there is no z such that $x < z < y$). Thus,

$$t \models \bigcirc\phi \quad \text{iff} \quad \forall_{t' \in \{p' \mid t \lessdot t'\}} \, . \, t' \models \phi$$

$$t \models \Box\phi \quad \text{iff} \quad \forall_{t' \in \{p' \mid t < t'\}} \, . \, t' \models \phi$$

$$t \models \Diamond\phi \quad \text{iff} \quad \exists_{t' \in \{p' \mid t < t'\}} \, . \, t' \models \phi$$

Examples III

... but typical structures, however, are

Linear time structures

- linear: $\langle \forall\ x, y\ :\ x, y \in T :\ x = y\ \vee\ x < y\ \vee\ y < x \rangle$.
- discrete: linear and for each $t \in T$,
  $(\exists u \cdot u > t)\ \Rightarrow\ \exists u' > t$ without any v s.t. $u' > v > t$ (and its dual)
- dense: if for all $t, x \in T$, if $x < t$ there is a $v \in T$ such that $x < v < t$.
- Dedekind complete: if for all $S \subseteq T$ non-empty and bounded above, there is a least upper bound in T.
- continuous: if it is both dense and Dedekind complete

Examples IV

Epistemic logic (J. Hintikka, 1962)

- W is a set of agents
- $\alpha \models i$ means i is the current knowledge of agent i
- $\alpha \models \Box j$ means the agent knows that j (in the sense that at each alternative epistemic situation information j is known)
- $\alpha \models \Diamond j$ means the agent knows that knowledge j is consistent with what the agent knows (is an epistemically acceptable alternative)

The first order connection

From modal logic

$$\phi \;::=\; p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \to \phi_2 \mid \langle m\rangle\,\phi \mid [m]\,\phi$$

To first order logic

$$\phi \;::=\; P\,x \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1\wedge\phi_2 \mid \phi_1\to\phi_2 \mid \langle\exists\; x \;::\; \phi\rangle \mid \langle\forall\; y \;::\; \phi\rangle$$

The first order connection

Boxes and diamonds are essentially a macro notation to encode quantification over accessible states in a point free way.

The standard translation

... to first-order logic expands these macros:

$$
\begin{aligned}
ST_x(p) &= P\,x \\
ST_x(\mathrm{true}) &= \mathrm{true} \\
ST_x(\mathrm{false}) &= \mathrm{false} \\
ST_x(\neg\phi) &= \neg ST_x(\phi) \\
ST_x(\phi_1 \wedge \phi_2) &= ST_x(\phi_1) \wedge ST_x(\phi_2) \\
ST_x(\phi_1 \rightarrow \phi_2) &= ST_x(\phi_1) \rightarrow ST_x(\phi_2) \\
ST_x(\langle m\rangle\,\phi) &= \langle \exists\ y\ ::\ (xR_m y\ \wedge\ ST_y(\phi))\rangle \\
ST_x([m]\,\phi) &= \langle \forall\ y\ ::\ (xR_m y\ \rightarrow\ ST_y(\phi))\rangle
\end{aligned}
$$

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ST_x(\langle m \rangle\,\phi) &= \langle \exists\ y \ :: \ (xR_m y \ \wedge \ ST_y(\phi)) \rangle \\
ST_x([m]\,\phi) &= \langle \forall\ y \ :: \ (xR_m y \ \rightarrow \ ST_y(\phi)) \rangle
\end{aligned}
$$

Translate: $ST_x(p \rightarrow \Diamond p)$

The first order connection

Lemma

For any ϕ, $\mathfrak{M}$ and point w in $\mathfrak{M}$,

$$\mathfrak{M}, w \models \phi \quad \text{iff} \quad \mathfrak{M} \models ST_x(\phi)[x \leftarrow w]$$

Note

Note how the (unique) free variable x in ST_x mirrors in first-order the internal perspective: assigning a value to x corresponds to evaluating the modal formula at a certain state.

The first order connection

The standard translation provides a bridge between modal logic and classical logic which makes possible to transfer results from one side to the other. For example,

Compactness

If Φ is a set of basic modal formulas and every finite subset of Φ is satisfiable, then Φ itself is satisfiable.

Löwenheim-Skolem

If Φ is a set of basic modal formulas satisfiable in at least one infinite model, then it is satisfiable in models of every infinite cardinality.

Summing up

- Propositional modal languages are syntactically simple languages that offer a pointfree notation for talking about relational structures
- They do this from the inside, using the modal operators to look for information at accessible states
- Regarded as a tool for talking about models, any basic modal language can be seen as a fragment of first-order language
- The standard translation systematically maps modal formulas to first-order formulas (in one free variable) and makes the quantification over accessible states explicit

Exercise

Express the following properties in Process Logic

- inevitability of a:
- progress:
- deadlock or termination:

"$-$" stands for Act, and "$-x$" abbreviates $Act - \{x\}$

Exercise

Express the following properties in Process Logic

- inevitability of a: $\langle - \rangle \, \mathsf{true} \wedge [-a] \, \mathsf{false}$
- progress:
- deadlock or termination:

"$-$" stands for Act, and "$-x$" abbreviates $Act - \{x\}$

Exercise

Express the following properties in Process Logic

- inevitability of a: $\langle - \rangle\, \mathsf{true} \wedge [-a]\, \mathsf{false}$
- progress: $\langle - \rangle\, \mathsf{true}$
- deadlock or termination:

"$-$" stands for Act, and "$-x$" abbreviates $Act - \{x\}$

Exercise

Express the following properties in Process Logic

- inevitability of a: $\langle - \rangle \, \mathsf{true} \wedge [-a] \, \mathsf{false}$
- progress: $\langle - \rangle \, \mathsf{true}$
- deadlock or termination: $[-] \, \mathsf{false}$
- what about

$$\langle - \rangle \, \mathsf{false} \quad \text{and} \quad [-] \, \mathsf{true} \quad ?$$

"$-$" stands for Act, and "$-x$" abbreviates $Act - \{x\}$

Exercise

Express the following properties in Process Logic

- $\phi_0 =$ *In a taxi network, a car can collect a passenger or be allocated by the Central to a pending service*
- $\phi_1 =$ *This applies only to cars already on-service*
- $\phi_2 =$ *If a car is allocated to a service, it must first collect the passenger and then plan the route*
- $\phi_3 =$ *On detecting an emergence the taxi becomes inactive*
- $\phi_4 =$ *A car on-service is not inactive*

Exercise

Process logic: The taxi network example

- $\phi_0 = \langle rec, alo \rangle \, \mathsf{true}$
- $\phi_1 = [onservice] \, \langle rec, alo \rangle \, \mathsf{true}$ or
  $\phi_1 = [onservice] \, \phi_0$
- $\phi_2 = [alo] \, \langle rec \rangle \, \langle plan \rangle \, \mathsf{true}$
- $\phi_3 = [sos] \, [-] \, \mathsf{false}$
- $\phi_4 = [onservice] \, \langle - \rangle \, \mathsf{true}$

Exercise

Standard translation to FOL

- Explain how propositional symbols and modalities are translated to first-order logic?
- In what sense can modal logic be regarded as a pointfree version of a FOL fragment?
- Compute $ST_x(p \Rightarrow \langle m \rangle\, p)$

Bisimulation (of models)

Definition

Given two models $\mathfrak{M} = \langle\langle W, R\rangle, V\rangle$ and $\mathfrak{M}' = \langle\langle W', R'\rangle, V'\rangle$, a bisimulation is a non-empty binary relation $S \subseteq W \times W'$ st whenever wSw' one has that

1. points w and w' satisfy the same propositional symbols
2. if wRv, then there is a point v' in $\mathfrak{M}'$ st $w'R'v'$ and vSv' (zig)
3. if $w'R'v'$, then there is a point v in $\mathfrak{M}$ st wRv and vSv' (zag)

Invariance and definability

Lemma (invariance: bisimulation implies modal equivalence)

Given two models $\mathfrak{M} = \langle\langle W, R\rangle, V\rangle$ and $\mathfrak{M}' = \langle\langle W', R'\rangle, V'\rangle$, and a bisimulation $S \subseteq W \times W'$,

if two points w, w' are related by S (i.e. wSw'),

then w, w' satisfy the same basic modal formulas.

(i.e., for all ϕ: $\mathfrak{M}, w \models \phi \Leftrightarrow \mathfrak{M}', w' \models \phi$)

Applications

- to prove bisimulation failures
- to show the undefinability of some structural notions, e.g. irreflexivity is modally undefinable
- to show that typical model constructions are satisfaction preserving
- ...

Exercise

Find characterising formulas

$$
\begin{tikzcd}
& 2 \arrow[loop above] \arrow[dr] & \\
1 \arrow[ur] \arrow[rr] \arrow[dr] & & 4 \\
& 3 \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& 6 \arrow[dr] & \\
5 \arrow[ur] \arrow[rr] \arrow[dr] & & 8 \\
& 7 \arrow[ur] &
\end{tikzcd}
$$

e.g., (4) is the only world satisfying $\Box\bot$

Frame definability

- A modal formula is valid on a frame if it is true under every valuation at every world (i.e., it cannot be refuted)
- The class of frames defined by a modal formula ϕ are those where ϕ is valid.
- Example: $\Diamond\Diamond p \to \Diamond p$ defines transitivity:
  $\mathfrak{F} = \langle W, R \rangle$ is transitive iff for all V and w,
  $\langle \mathfrak{F}, V \rangle, w \models \Diamond\Diamond p \to \Diamond p$

Exercise

Exercise: other properties

1. Transitivity: $\Diamond\Diamond p \to \Diamond p$
2. Reflexivity:
3. Symmetry:
4. Confluence:
5. Irreflexibility:

Exercise

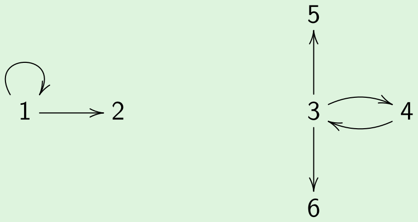
Exercise: other properties

1. Transitivity: $\Diamond\Diamond p \to \Diamond p$
2. Reflexivity: $p \to \Diamond p$
3. Symmetry: $p \to \Box\Diamond p$
4. Confluence: $\Diamond \Box p \to \Box\Diamond p$
5. Irreflexibility: Not possible

Exercise

Bisimilarity and modal equivalence

- Consider the following transition systems:



  Give a modal formula that can be satisfied at point 1 but not at 3.

- Show that irreflexivity is modally undefinable.
  (*i.e.*, no formula that characterises a irreflexive system)

- Prove the invariance lemma.

Invariance and definability

To prove the converse of the invariance lemma requires passing to an infinitary modal language with arbitrary (countable) conjunctions and disjunctions. Alternatively, and more usefully, it can be shown for finite models:

Lemma (modal equivalence implies bisimulation)

If two points w, w' from two finite models $\mathfrak{M} = \langle\langle W, R\rangle, V\rangle$ and $\mathfrak{M}' = \langle\langle W', R'\rangle, V'\rangle$ satisfy the same modal formulas, then there is a bisimulation $S \subseteq W \times W'$ such that wSw'.

Invariance and definability

Note

- The result can be weakened to image-finite models.
- Combining this result with the invariance lemma one gets the so-called modal equivalence theorem stating that, for image-finite models, bisimilarity and modal equivalence coincide. The result is also known as the Hennessy-Milner theorem who first proved it for process logics.

Exercise

- Give an example of modally equivalent states in different Kripke structures which fail to be bisimilar.

Invariance and definability

Lemma (modal logic vs first-order)

The following are equivalent for all first-order formulas $\phi(x)$ in one free variable x:

1. $\phi(x)$ is invariant for bisimulation.
2. $\phi(x)$ is equivalent to the standard translation of a basic modal formula.

Therefore:
the basic modal language corresponds to the fragment of their first-order correspondence language that is invariant for bisimulation

Invariance and definability

- the basic modal language (interpreted over the class of all models) is computationally better behaved than the corresponding first-order language (interpreted over the same models)
- ... but clearly less expressive

	model checking	satisfiability
ML	PTIME	PSPACE-complete
FOL	PSPACE-complete	undecidable

What are the trade-offs? Can this better computational behaviour be lifted to more expressive modal logics?

mCRL2 - modal logic

Syntax (simplified)

$$\phi = \texttt{true} \mid \texttt{false} \mid \texttt{forall x:T.}\phi \mid \texttt{exists x.:T}\phi$$
$$\mid \phi \; OP \; \phi \mid \texttt{!}\phi \mid \texttt{[}mod\texttt{]}\phi \mid \texttt{<}mod\texttt{>}\phi \mid \ldots$$

$$mod = \alpha \mid \texttt{nil} \mid mod\texttt{+}mod \mid mod\texttt{.}mod \mid mod* \mid mod\texttt{+}$$

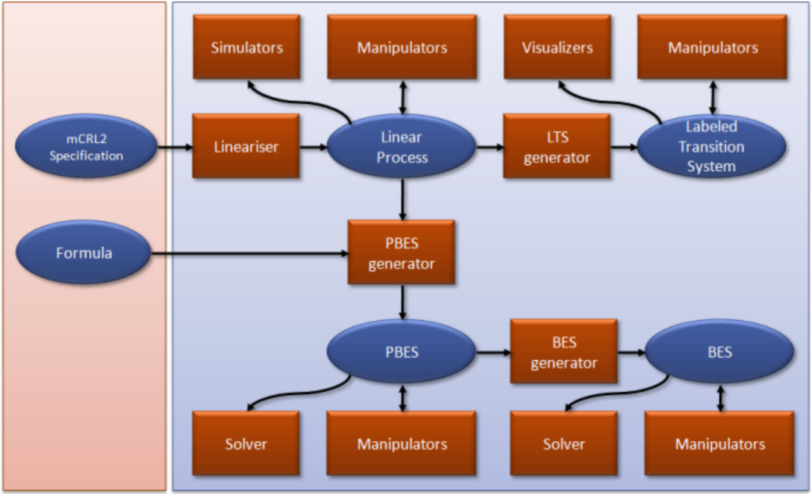
$$\alpha = \texttt{a(d)} \mid \texttt{a|b|c} \mid \texttt{true} \mid \texttt{false} \mid \alpha \; OP \; \alpha \mid \texttt{!}\alpha$$
$$\mid \texttt{forall x:T.}\alpha \mid \texttt{exists x:T.}\alpha \mid \ldots$$

where $OP = \{\texttt{\&\&}, \texttt{||}, \texttt{=>}\}$ and $T = \{Bool, Nat, Int, \ldots\}$

Example

"`[true*.a]<b>true`" means *"whenever an a appears after any number of steps, it must be immediately followed by* `b`*"*.

mCRL2 toolset overview



– mCRL2 tutorial: Verification part –

Richer modal logics

can be obtained in different ways, e.g.

- axiomatic extensions
- introducing more complex satisfaction relations
- support novel semantic capabilities
- ...

Examples

- richer temporal logics
- hybrid logic
- modal μ-calculus

Temporal Logics with $\mathcal{U}$ and $\mathcal{S}$

Until and Since

$$\mathfrak{M}, w \models \phi\,\mathcal{U}\,\psi \quad \text{iff} \quad \text{there exists } v \text{ st } w \leq v \text{ and } \mathfrak{M}, v \models \psi\text{, and for all } u \text{ st } w \leq u < v\text{, one has } \mathfrak{M}, u \models \phi$$

$$\mathfrak{M}, w \models \phi\,\mathcal{S}\,\psi \quad \text{iff} \quad \text{there exists } v \text{ st } v \leq w \text{ and } \mathfrak{M}, v \models \psi\text{, and for all } u \text{ st } v < u \leq w\text{, one has } \mathfrak{M}, u \models \phi$$

- Defined for temporal frames $\langle T, < \rangle$ (transitive, asymmetric).
- note the $\exists\forall$ qualification pattern: these operators are neither diamonds nor boxes.
- More general definition for other frames – it becomes more expressive than modal logics.

Exercise

Temporal logics - rewrite using $\mathcal{U}$

- $\Diamond\psi =$
- $\Box\psi =$

Exercise

Temporal logics - rewrite using $\mathcal{U}$

- $\Diamond\psi = tt\,\mathcal{U}\,\psi$
- $\Box\psi =$

Exercise

Temporal logics - rewrite using $\mathcal{U}$

- $\Diamond\psi = tt\,\mathcal{U}\,\psi$
- $\Box\psi = \neg(\Diamond\neg\psi) = \neg(tt\,\mathcal{U}\,\neg\psi)$

Linear temporal logic (LTL)

$$\phi ::= \mathsf{true} \mid p \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \bigcirc\phi \mid \phi_1 \,\mathcal{U}\, \phi_2$$

mutual exclusion	$\Box(\neg c_1 \vee \neg c_2)$
liveness	$\Box\Diamond c_1 \wedge \Box\Diamond c_2$
starvation freedom	$(\Box\Diamond w_1 \to \Box\Diamond c_1) \wedge (\Box\Diamond w_1 \to \Box\Diamond c_1)$
progress	$\Box(w_1 \to \Diamond c_1)$
weak fairness	$\Diamond\Box w_1 \to \Box\Diamond c_1$
eventually forever	$\Diamond\Box w_1$

- First temporal logic to reason about reactive systems [Pnueli, 1977]
- Formulas are interpreted over execution paths
- Express linear-time properties

Computational tree logic (CTL, CTL*)

state formulas to express properties of a state:

$$\Phi \ := \ \mathsf{true} \mid \Phi \wedge \Phi \mid \neg\Phi \mid \exists\psi \mid \forall\psi$$

path formulas to express properties of a path:

$$\psi \ := \ \bigcirc\Phi \mid \Phi\,\mathcal{U}\,\Psi$$

mutual exclusion	$\forall\Box(\neg c_1 \vee \neg c_2)$
liveness	$\forall\Box\forall\Diamond c_1 \wedge \forall\Box\forall\Diamond c_2$
order	$\forall\Box(c_1 \vee \forall\bigcirc c_2)$

- Branching time structure encode transitive, irreflexive but not necessarily linear flows of time
- flows are trees: past linear; branching future

Hybrid logic

Motivation

Add the possibility of naming points and reason about their identity

Compare:

$$\Diamond(r \wedge p) \ \wedge\ \Diamond(r \wedge q) \ \rightarrow\ \Diamond(p \wedge q)$$

with

$$\Diamond(i \wedge p) \ \wedge\ \Diamond(i \wedge q) \ \rightarrow\ \Diamond(p \wedge q)$$

for $i \in \mathrm{NOM}$ (a nominal)

Syntax

$$\phi \ ::= \ \ldots \mid p \mid \langle m \rangle \phi \mid [m] \phi \mid i \mid @_i \phi$$

where $p \in \mathrm{PROP}$ and $m \in \mathrm{MOD}$ and $i \in \mathrm{NOM}$

Hybrid logic

Nominals i

- Are special propositional symbols that hold exactly on one state (the state they name)
- In a model the valuation V is extended from

$$V : \mathrm{PROP} \longrightarrow \mathcal{P}(W)$$

to

$$V : \mathrm{PROP} \longrightarrow \mathcal{P}(W) \quad \text{and} \quad V : \mathrm{NOM} \longrightarrow W$$

where NOM is the set of nominals in the model

- Satisfaction:

$$\mathfrak{M}, w \models i \qquad \text{iff } w = V(i)$$

Hybrid logic

The $@_i$ operator

$\mathfrak{M}, w \models @_i\phi$ iff $\mathfrak{M}, u \models \phi$ and $u = V(i)$ [u is the state denoted by i]

Standard translation to first-order

$$ST_x(i) = (x = i)$$
$$ST_x(@_i\phi) = ST_i(\phi)[x \leftarrow i]$$

i.e., hybrid logic corresponds to a first-order language enriched with constants and equality.

Hybrid logic

Increased frame definability

- irreflexivity: $i \to \neg\Diamond i$
- asymmetry: $i \to \neg\Diamond\Diamond i$
- antisymmetry: $i \to \Box(\Diamond i \to i)$
- trichotomy: $@_j \Diamond i \ \vee \ @_i j \ \vee \ @_i \Diamond j$

Hybrid logic

Summing up

- basic hybrid logic is a simple notation for capturing the bisimulation-invariant fragment of first-order logic with constants and equality, i.e., a mechanism for equality reasoning in propositional modal logic.
- comes cheap: up to a polynomial, the complexity of the resulting decision problem is no worse than for the basic modal language