A brief Introduction to Category Theory

Dirk Hofmann

CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
Office: 11.3.10, dirk@ua.pt, http://sweet.ua.pt/dirk/

October 9, 2017

Motivation I

John Hughes (1989). "Why functional programming matters". In: *The Computer Journal* **32**.(2), pp. 98–107

Modular design is the key to successful programming

. . . The ways in which one can divide up the original problem depend directly on the ways in which one can glue solutions together. Therefore, to increase ones ability to modularise a problem conceptually, one must provide new kinds of glue in the programming language.

. . .

Now let us return to functional programming. We shall argue in the remainder of this paper that functional languages provide two new, very important kinds of glue.

Motivation I

Whitehead, Alfred North

Mathematics as a science, commenced when first someone, probably a Greek, proved propositions about "any" things or about "some" things, without specifications of definite particular things.

Motivation I

Whitehead, Alfred North

Mathematics as a science, commenced when first someone, probably a Greek, proved propositions about "any" things or about "some" things, without specifications of definite particular things.

Saunders MacLane

The basic insight is that a mathematical structure is a scientific structure but one which has many different empirical realizations. Mathematics provides common overreaching forms, each of which can and does serve to describe different aspects of the external world. Thus mathematics is that part of science which applies in more than one empirical context.

Saunders MacLane (1997). "Despite physicist, proof is essential in mathematics". In: *Synthese* **111**.(2), pp. 147–154.

Motivation II

A seemingly paradoxical observation

"...an equation is only interesting or useful to the extent that the two sides are different!"

John Baez and James Dolan (2001). "From finite sets to Feynman diagrams". In: *Mathematics Unlimited - 2001 and Beyond*. Springer, Berlin, pp. 29–50

Motivation II

A seemingly paradoxical observation

"...an equation is only interesting or useful to the extent that the two sides are different!"

John Baez and James Dolan (2001). "From finite sets to Feynman diagrams". In: *Mathematics Unlimited - 2001 and Beyond*. Springer, Berlin, pp. 29–50

Just compare:

- Numbers: $3 = 3$ vs. $e^{i\omega} = \cos(\omega) + i\sin(\omega)$.

Motivation II

A seemingly paradoxical observation

"...an equation is only interesting or useful to the extent that the two sides are different!"

John Baez and James Dolan (2001). "From finite sets to Feynman diagrams". In: *Mathematics Unlimited - 2001 and Beyond*. Springer, Berlin, pp. 29–50

Just compare:

- Numbers: $3 = 3$ vs. $e^{i\omega} = \cos(\omega) + i\sin(\omega)$.
- Spaces: $V \simeq V$ vs. $V \simeq \mathbb{R}^n$ (for $\dim V = n$).

Motivation II

A seemingly paradoxical observation

"…an equation is only interesting or useful to the extent that the two sides are different!"

John Baez and James Dolan (2001). "From finite sets to Feynman diagrams". In: *Mathematics Unlimited - 2001 and Beyond*. Springer, Berlin, pp. 29–50

Just compare:

- Numbers: $3 = 3$ vs. $e^{i\omega} = \cos(\omega) + i\sin(\omega)$.
- Spaces: $V \simeq V$ vs. $V \simeq \mathbb{R}^n$ (for $\dim V = n$).
- More general: "linear maps = matrices".

A comercial break

Vladimir Voevodsky (Fields Medal in 2002)

"At the heart of 20th century mathematics lies one particular notion, and that is the notion of a category."

A comercial break

Vladimir Voevodsky (Fields Medal in 2002)

"At the heart of 20th century mathematics lies one particular notion, and that is the notion of a category."

Jean Bénabou

"Analogies are useful in mathematics for generalising a class of well-known examples to a wider class of equally or even more useful structures. Category Theory is particularly well suited for this purpose which is no wonder as it has been developed for precisely this purpose"

A comercial break

Vladimir Voevodsky (Fields Medal in 2002)

"At the heart of 20th century mathematics lies one particular notion, and that is the notion of a category."

Jean Bénabou

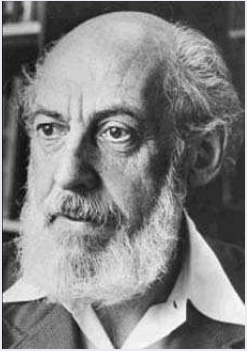
"Analogies are useful in mathematics for generalising a class of well-known examples to a wider class of equally or even more useful structures. Category Theory is particularly well suited for this purpose which is no wonder as it has been developed for precisely this purpose"

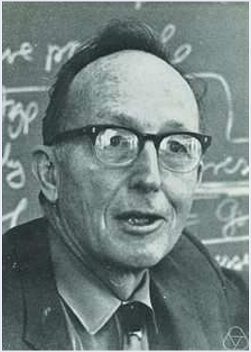
Bill Lawvere

"The kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' . . . , as concentrated in the thesis that *fundamental* structures are themselves categories."

Category Theory

Sammy Eilenberg (1913 – 1998) and Saunders MacLane (1909 – 2005)

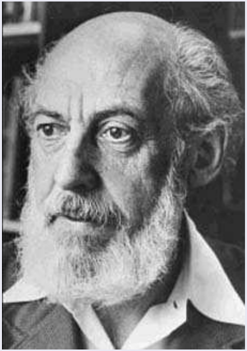


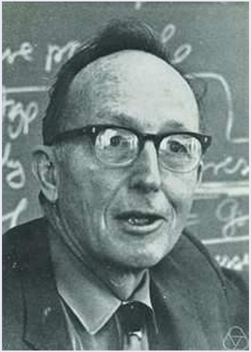


- Started in the 1940's in their work about algebraic topology.

Category Theory

Sammy Eilenberg (1913 – 1998) and Saunders MacLane (1909 – 2005)





- Started in the 1940's in their work about algebraic topology.
- Is by now present in (almost) all areas of mathematics and also extensively used in physics and in computer science.

Bibliography

- Saunders MacLane (1998). *Categories for the working mathematician*. 2nd ed. New York: Springer-Verlag, pp. ix + 262. Graduate Texts in Mathematics, Vol. 5.
- Tom Leinster (2014a). *Basic Category Theory*. Cambridge University Press. 190 pp.
- Tom Leinster (2014b). "Rethinking set theory". In: *American Mathematical Monthly* **121**.(5), pp. 403–415.
- F. William Lawvere and Robert Rosebrugh (2003). *Sets for mathematics*. English. Cambridge: Cambridge University Press, pp. xi + 261.
- F. William Lawvere and Stephen H. Schanuel (2009). *Conceptual mathematics. A first introduction to categories*. English. 2nd ed. Cambridge University Press, pp. xii + 390.
- Andrea Asperti and Giuseppe Longo (1991). *Categories, types, and structures*. Foundations of Computing Series. Cambridge, MA: MIT Press, pp. xii+306. An introduction to category theory for the working computer scientist.

So what is it about?

So what is it about?

Definition

A category $\mathbf{X}$ consists of

So what is it about?

Definition

A category $\mathbf{X}$ consists of

- a collection of objects X, Y, $\ldots$,

Think of

- vector spaces, ordered sets, topological spaces, . . .

So what is it about?

Definition

A category **X** consists of

- a collection of objects X, Y, ...,
- arrows (morphisms) $f\colon X \to Y$ between objects,

Think of

- vector spaces, ordered sets, topological spaces, ...
- linear maps, monotone maps, continuous maps, ...

So what is it about?

Definition

A category $\mathbf{X}$ consists of

- a collection of objects $X, Y, \ldots,$
- arrows (morphisms) $f\colon X \to Y$ between objects,
- arrows can be composed (associativity)

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad X \xrightarrow{g \cdot f} Z$$

Think of

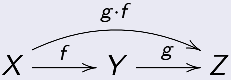
- vector spaces, ordered sets, topological spaces, ...
- linear maps, monotone maps, continuous maps, ...
- the composite of linear maps is linear, ...

So what is it about?

Definition

A category $\mathbf{X}$ consists of

- a collection of objects X, Y, . . . ,
- arrows (morphisms) $f : X \to Y$ between objects,
- arrows can be composed (associativity)



- for every object there is an identity arrow $1_X : X \to X$.

Think of

- vector spaces, ordered sets, topological spaces, . . .
- linear maps, monotone maps, continuous maps, . . .
- the composite of linear maps is linear, . . .
- The identity map is linear, ...

So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, $\mathbf{Vec}_{\text{fin}}$,

So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, $\mathbf{Vec}_{\mathrm{fin}}$, **Ban**,

So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, $\textbf{Vec}_{\mathrm{fin}}$, **Ban**, **Met**,

So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, $\mathbf{Vec}_{\text{fin}}$, **Ban**, **Met**, **Met**, . . .

So what is it about?

Every field of mathematics defines (at least) one category

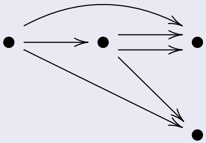
Top, **Grp**, $\mathbf{Vec}_{\mathrm{fin}}$, **Ban**, **Met**, **Met**, ..., **Rel**, **Mat** ...

So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, **Vec**$_{\text{fin}}$, **Ban**, **Met**, **Met**, . . . , **Rel**, **Mat** . . .

An abstract category . . .

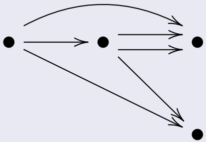


So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, $\mathbf{Vec}_{\mathrm{fin}}$, **Ban**, **Met**, **Met**, . . . , **Rel**, **Mat** . . .

An abstract category . . .



Definition

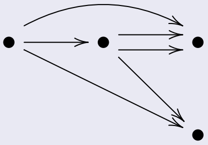
For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\mathrm{op}}$ with the same objects but all arrows point in the opposite direction.

So what is it about?

Every field of mathematics defines (at least) one category

Top, **Grp**, $\mathbf{Vec}_{\text{fin}}$, **Ban**, **Met**, **Met**, ..., **Rel**, **Mat** ...

An abstract category ...



Definition

For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\text{op}}$ with the same objects but all arrows point in the opposite direction.

Examples

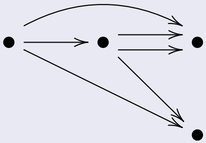
$\mathbf{Top}^{\text{op}}$, $\mathbf{Grp}^{\text{op}}$, $\mathbf{Vec}_{\text{fin}}^{\text{op}}$, ..., $\mathbf{Rel}^{\text{op}}$, $\mathbf{Mat}^{\text{op}}$ ...

So what is it about?

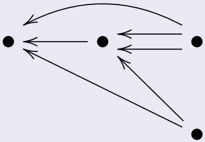
Every field of mathematics defines (at least) one category

Top, **Grp**, $\mathbf{Vec}_{\mathrm{fin}}$, **Ban**, **Met**, **Met**, . . . , **Rel**, **Mat** . . .

An abstract category . . .



. . . and its dual



Definition

For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\mathrm{op}}$ with the same objects but all arrows point in the opposite direction.

Examples

$\mathbf{Top}^{\mathrm{op}}$, $\mathbf{Grp}^{\mathrm{op}}$, $\mathbf{Vec}_{\mathrm{fin}}^{\mathrm{op}}$, . . . , $\mathbf{Rel}^{\mathrm{op}}$, $\mathbf{Mat}^{\mathrm{op}}$ . . .

Some typical categorical notions

Some typical categorical notions

Isomorphism

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with

$$g \cdot f = 1_X \qquad \text{and} \qquad f \cdot g = 1_Y$$

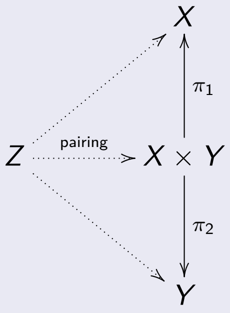
Some typical categorical notions

Isomorphism

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with

$$g \cdot f = 1_X \qquad \text{and} \qquad f \cdot g = 1_Y$$

Product in $\mathbf{X}$



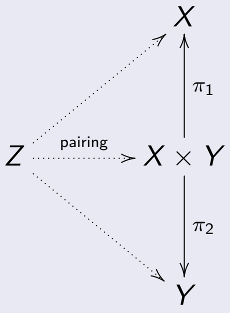
Some typical categorical notions

Isomorphism

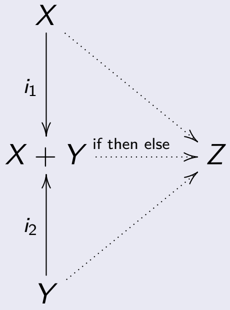
An arrow $f : X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g : Y \to X$ with

$$g \cdot f = 1_X \qquad \text{and} \qquad f \cdot g = 1_Y$$

Product in $\mathbf{X}$



Sum in $\mathbf{X}$



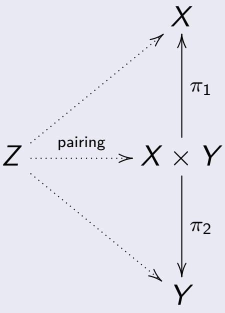
Some typical categorical notions

Isomorphism

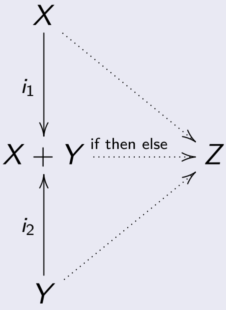
An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with

$$g \cdot f = 1_X \qquad \text{and} \qquad f \cdot g = 1_Y$$

Product in $\mathbf{X}$



Sum in $\mathbf{X}$ = product in $\mathbf{X}^{\mathrm{op}}$



When are two categories “equal”?

Equivalence $\mathbf{X} \sim \mathbf{Y}$ of categories

When are two categories “equal”?

Equivalence $\mathbf{X} \sim \mathbf{Y}$ of categories

- A functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$:

$$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.

When are two categories "equal"?

Equivalence $\mathbf{X} \sim \mathbf{Y}$ of categories

- A functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$:

$$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

  so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.
- A functor $G\colon \mathbf{Y} \longrightarrow \mathbf{X}$.

When are two categories "equal"?

Equivalence $\mathbf{X} \sim \mathbf{Y}$ of categories

- A functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$:

$$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

  so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.
- A functor $G\colon \mathbf{Y} \longrightarrow \mathbf{X}$.
- Natural isomorphisms $\eta_X\colon X \to GFX$ and $\varepsilon_Y : FGY \to Y$ ...

When are two categories "equal"?

Equivalence $\mathbf{X} \sim \mathbf{Y}$ of categories

- A functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$:

$$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

  so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.
- A functor $G\colon \mathbf{Y} \longrightarrow \mathbf{X}$.
- Natural isomorphisms $\eta_X\colon X \to GFX$ and $\varepsilon_Y : FGY \to Y$ ...

Adjunction

As above but the arrows $\eta_X\colon X \to GFX$ and $\varepsilon_Y : FGY \to Y$ need not be isomorphisms ...

When are two categories "equal"?

Equivalence $\mathbf{X} \sim \mathbf{Y}$ of categories

- A functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$:

$$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

  so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.
- A functor $G\colon \mathbf{Y} \longrightarrow \mathbf{X}$.
- Natural isomorphisms $\eta_X\colon X \to GFX$ and $\varepsilon_Y : FGY \to Y$ ...

Adjunction

As above but the arrows $\eta_X\colon X \to GFX$ and $\varepsilon_Y : FGY \to Y$ need not be isomorphisms ...

Definition

Let $F, G\colon \mathbf{X} \longrightarrow \mathbf{Y}$ be functors. An natural transformation α is a family $(\alpha_X\colon FX \to GX)_X$ which commutes with arrows in $\mathbf{X}$.

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\text{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)
- **Theorem.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Mat}$.

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)
- **Theorem.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Mat}$.
  - $F\colon \mathbf{Mat} \longrightarrow \mathbf{Vec}_{\mathrm{fin}},\ (A\colon n \to m) \longmapsto (f_A\colon \mathbb{R}^n \to \mathbb{R}^m)$.

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)
- **Theorem.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Mat}$.
  - $F\colon \mathbf{Mat} \longrightarrow \mathbf{Vec}_{\mathrm{fin}}$, $(A\colon n \to m) \longmapsto (f_A\colon \mathbb{R}^n \to \mathbb{R}^m)$.
  - $G\colon \mathbf{Vec}_{\mathrm{fin}} \longrightarrow \mathbf{Mat}$,
    $(f\colon V \to W) \longmapsto (\text{matrix of f}\colon \dim V \to \dim W)$.
    Requires choosing a base for every space.

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)
- **Theorem.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Mat}$.
  - $F\colon \mathbf{Mat} \longrightarrow \mathbf{Vec}_{\mathrm{fin}}$, $(A\colon n \to m) \longmapsto (f_A\colon \mathbb{R}^n \to \mathbb{R}^m)$.
  - $G\colon \mathbf{Vec}_{\mathrm{fin}} \longrightarrow \mathbf{Mat}$,
    $(f\colon V \to W) \longmapsto (\text{matrix of f}\colon \dim V \to \dim W)$.
    Requires choosing a base for every space.
  - $n = \dim \mathbb{R}^n$ and $V \simeq \mathbb{R}^n$

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)
- **Theorem.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Mat}$.
  - $F\colon \mathbf{Mat} \longrightarrow \mathbf{Vec}_{\mathrm{fin}},\ (A\colon n \to m) \longmapsto (f_A\colon \mathbb{R}^n \to \mathbb{R}^m)$.
  - $G\colon \mathbf{Vec}_{\mathrm{fin}} \longrightarrow \mathbf{Mat}$,
    $(f\colon V \to W) \longmapsto (\text{matrix of f}\colon \dim V \to \dim W)$.
    Requires choosing a base for every space.
  - $n = \dim \mathbb{R}^n$ and $V \simeq \mathbb{R}^n$
- **Theorem.** $\mathbf{Mat} \sim \mathbf{Mat}^{\mathrm{op}}$.
  Here: $(A\colon n \to m) \longmapsto (A^T\colon m \to n)$, $(B \cdot A)^T = A^T \cdot B^T$; $I^T = I$.

An elementary example

Linear algebra via matrices

- Let $\mathbf{X} = \mathbf{Vec}_{\mathrm{fin}}$ be the category of finite dimensional vector spaces (over $\mathbb{R}$) and linear maps.
- Let $\mathbf{Y} = \mathbf{Mat}$ be the category of natural numbers and matrices. (Here $A\colon n \to m$ means A is a matrix of type $m \times n$.)
- **Theorem.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Mat}$.
  - $F: \mathbf{Mat} \longrightarrow \mathbf{Vec}_{\mathrm{fin}}, \ (A\colon n \to m) \longmapsto (f_A : \mathbb{R}^n \to \mathbb{R}^m)$.
  - $G: \mathbf{Vec}_{\mathrm{fin}} \longrightarrow \mathbf{Mat}$,
    $(f: V \to W) \longmapsto (\text{matrix of f}\colon \dim V \to \dim W)$.
    Requires choosing a base for every space.
  - $n = \mathrm{dim}\mathbb{R}^n$ and $V \simeq \mathbb{R}^n$
- **Theorem.** $\mathbf{Mat} \sim \mathbf{Mat}^{\mathrm{op}}$.
  Here: $(A\colon n \to m) \longmapsto (A^T : m \to n)$, $(B \cdot A)^T = A^T \cdot B^T$; $I^T = I$.
- **Corollary.** $\mathbf{Vec}_{\mathrm{fin}} \sim \mathbf{Vec}_{\mathrm{fin}}^{\mathrm{op}}$.

A brief Introduction to Category Theory
Aula 2

Dirk Hofmann

CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
Office: 11.3.10, dirk@ua.pt, http://sweet.ua.pt/dirk/

October 16, 2017

Recall from last week

1. A category **X** consists of
   - a collection of objects X, Y, $\ldots$;
   - for each pair of objects, a *set* of arrows (morphisms) $f : X \to Y$ (denoted as $\mathbf{X}(X, Y)$ or $\hom(X, Y)$);
   - arrows can be composed (associativity) and for every object there is an identity arrow $1_X : X \to Y$.

Recall from last week

1. A category $\mathbf{X}$ consists of
   - a collection of objects X, Y, $\ldots$;
   - for each pair of objects, a *set* of arrows (morphisms) $f\colon X \to Y$ (denoted as $\mathbf{X}(X, Y)$ or $\hom(X, Y)$);
   - arrows can be composed (associativity) and for every object there is an identity arrow $1_X\colon X \to Y$.
2. For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\mathrm{op}}$ with the same objects but all arrows point in the opposite direction.

Recall from last week

1. A category $\mathbf{X}$ consists of
   - a collection of objects X, Y, ...;
   - for each pair of objects, a *set* of arrows (morphisms) $f : X \to Y$ (denoted as $\mathbf{X}(X, Y)$ or $\hom(X, Y)$);
   - arrows can be composed (associativity) and for every object there is an identity arrow $1_X : X \to Y$.
2. For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\mathrm{op}}$ with the same objects but all arrows point in the opposite direction.
3. Functor $F : \mathbf{X} \longrightarrow \mathbf{Y}$:

$$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

   so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.

Recall from last week

1. A category $\mathbf{X}$ consists of
   - a collection of objects $X, Y, \ldots$;
   - for each pair of objects, a *set* of arrows (morphisms) $f\colon X \to Y$ (denoted as $\mathbf{X}(X,Y)$ or $\hom(X,Y)$);
   - arrows can be composed (associativity) and for every object there is an identity arrow $1_X\colon X \to Y$.
2. For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\mathrm{op}}$ with the same objects but all arrows point in the opposite direction.
3. Functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$:

   $$(X_1 \xrightarrow{f} X_2) \longmapsto (FX_1 \xrightarrow{Ff} FX_2)$$

   so that $F(g \cdot f) = Fg \cdot Ff$ and $F1_X = 1_{FX}$.
4. Let $F, G\colon \mathbf{X} \longrightarrow \mathbf{Y}$ be functors. An natural transformation α is a family $(\alpha_X\colon FX \to GX)_X$ of $\mathbf{Y}$-arrows which commutes with arrows in $\mathbf{X}$.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category **X**:

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Monoids

Every monoid $(M, \cdot, e)$ can be viewed as a category $\mathbf{X}$:

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Monoids

Every monoid $(M, \cdot, e)$ can be viewed as a category $\mathbf{X}$:

- $\mathbf{X}$ has one object (denoted here as $\star$).

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Monoids

Every monoid $(M, \cdot, e)$ can be viewed as a category $\mathbf{X}$:

- $\mathbf{X}$ has one object (denoted here as $\star$).
- $\mathbf{X}(\star, \star) = M$.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Monoids

Every monoid $(M, \cdot, e)$ can be viewed as a category $\mathbf{X}$:

- $\mathbf{X}$ has one object (denoted here as $\star$).
- $\mathbf{X}(\star, \star) = M$.
- Composition of $\mathbf{X}$ = multiplication $\cdot$ of the monoid.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Monoids

Every monoid $(M, \cdot, e)$ can be viewed as a category $\mathbf{X}$:

- $\mathbf{X}$ has one object (denoted here as $\star$).
- $\mathbf{X}(\star, \star) = M$.
- Composition of $\mathbf{X}$ = multiplication $\cdot$ of the monoid.
- identity $e \colon \star \to \star$.

Two elementary examples

Ordered sets

Every ordered set $(X, \leq)$ can be viewed as a category $\mathbf{X}$:

- Objects of $\mathbf{X}$ = elements of X.
- There is an arrow $x \to y$ whenever $x \leq y$.
- Transitivity $\rightsquigarrow$ composition.
- Reflexivity $\rightsquigarrow$ identities.

Then: Functor means monotone map.

Monoids

Every monoid $(M, \cdot, e)$ can be viewed as a category $\mathbf{X}$:

- $\mathbf{X}$ has one object (denoted here as $\star$).
- $\mathbf{X}(\star, \star) = M$.
- Composition of $\mathbf{X}$ = multiplication $\cdot$ of the monoid.
- identity $e \colon \star \to \star$.

Then: Functor means monoid homomorphism.

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

Special morphisms

Definition

An arrow $f: X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g: Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f: X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.

Special morphisms

Definition

An arrow $f: X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g: Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f: X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism:
  exists $g: Y \to X$ with
  $g \cdot f = 1_X$.

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism:
  exists $g\colon Y \to X$ with
  $g \cdot f = 1_X$.
- f epimorphism:
  $h \cdot f = k \cdot f \implies h = k$.

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

- f monomorphism: $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism: exists $g\colon Y \to X$ with $g \cdot f = 1_X$.
- f epimorphism: $h \cdot f = k \cdot f \implies h = k$.
- f split epimorphism: exists $g\colon Y \to X$ with $f \cdot g = 1_Y$.

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism:
  exists $g\colon Y \to X$ with
  $g \cdot f = 1_X$.
- f epimorphism:
  $h \cdot f = k \cdot f \implies h = k$.
- f split epimorphism:
  exists $g\colon Y \to X$ with
  $f \cdot g = 1_Y$.

Some properties

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism:
  exists $g\colon Y \to X$ with
  $g \cdot f = 1_X$.
- f epimorphism:
  $h \cdot f = k \cdot f \implies h = k$.
- f split epimorphism:
  exists $g\colon Y \to X$ with
  $f \cdot g = 1_Y$.

Some properties

- split mono $\implies$ mono (and split epi $\implies$ epi).

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism:
  exists $g\colon Y \to X$ with
  $g \cdot f = 1_X$.
- f epimorphism:
  $h \cdot f = k \cdot f \implies h = k$.
- f split epimorphism:
  exists $g\colon Y \to X$ with
  $f \cdot g = 1_Y$.

Some properties

- split mono $\implies$ mono (and split epi $\implies$ epi).
- (split mono & epi) $\implies$ iso $\impliedby$ (split epi & mono).

Special morphisms

Definition

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with $g \cdot f = 1_X$ and $f \cdot g = 1_Y$.

More arrows $f\colon X \to Y$

- f monomorphism:
  $f \cdot h = f \cdot k \implies h = k$.
- f split monomorphism:
  exists $g\colon Y \to X$ with
  $g \cdot f = 1_X$.
- f epimorphism:
  $h \cdot f = k \cdot f \implies h = k$.
- f split epimorphism:
  exists $g\colon Y \to X$ with
  $f \cdot g = 1_Y$.

Some properties

- split mono $\implies$ mono (and split epi $\implies$ epi).
- (split mono & epi) $\implies$ iso $\impliedby$ (split epi & mono).
- Every functor preserves split monos/split epis/isos.

Limits and colimits

Some constructions in categories

- terminal object

Limits and colimits

Some constructions in categories

- terminal object
- product

Limits and colimits

Some constructions in categories

- terminal object
- product
- initial object
- sum

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- initial object
- sum

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback

- initial object
- sum
- pushout

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- initial object
- sum
- pushout

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser

- initial object
- sum
- pushout
- coequaliser

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit
- initial object
- sum
- pushout
- coequaliser

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit
- initial object
- sum
- pushout
- coequaliser
- colimit

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit
- initial object
- sum
- pushout
- coequaliser
- colimit

Definition

A category **X** is called (co)complete whenever **X** has all (co)limits.

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit

- initial object
- sum
- pushout
- coequaliser
- colimit

Definition

A category $\mathbf{X}$ is called (co)complete whenever $\mathbf{X}$ has all (co)limits.

Some facts

- Limit = terminal object in some category (uniqueness!!).

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit

- initial object
- sum
- pushout
- coequaliser
- colimit

Definition

A category $\mathbf{X}$ is called (co)complete whenever $\mathbf{X}$ has all (co)limits.

Some facts

- Limit $=$ terminal object in some category (uniqueness!!).
- $\mathbf{X}$ is complete $\iff$ $\mathbf{X}$ has products and equalisers.

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit

- initial object
- sum
- pushout
- coequaliser
- colimit

Definition

A category $\mathbf{X}$ is called (co)complete whenever $\mathbf{X}$ has all (co)limits.

Some facts

- Limit $=$ terminal object in some category (uniqueness!!).
- $\mathbf{X}$ is complete $\iff$ $\mathbf{X}$ has products and equalisers.
- $\mathbf{X}$ is finitely complete $\iff$ $\mathbf{X}$ has a terminal object and pullbacks.

Limits and colimits

Some constructions in categories

- terminal object
- product
- pullback
- equaliser
- limit
- initial object
- sum
- pushout
- coequaliser
- colimit

Definition

A category $\mathbf{X}$ is called (co)complete whenever $\mathbf{X}$ has all (co)limits.

Some facts

- Limit $=$ terminal object in some category (uniqueness!!).
- $\mathbf{X}$ is complete $\iff$ $\mathbf{X}$ has products and equalisers.
- $\mathbf{X}$ is finitely complete $\iff$ $\mathbf{X}$ has a terminal object and pullbacks.
- mono vs. pullback.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y$.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y$.

Definition

A functor $F : \mathbf{X} \to \mathbf{Set}$ is called representable whenever $F \simeq \mathbf{X}(X,-)$.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y$.

Definition

A functor $F : \mathbf{X} \to \mathbf{Set}$ is called representable whenever $F \simeq \mathbf{X}(X,-)$.

Examples

- For $F\colon \mathbf{Ord} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(1,-)$.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y$.

Definition

A functor $F : \mathbf{X} \to \mathbf{Set}$ is called representable whenever $F \simeq \mathbf{X}(X,-)$.

Examples

- For $F\colon \mathbf{Ord} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(1,-)$.
- For $F\colon \mathbf{Vec} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(\mathbb{R},-)$.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y$.

Definition

A functor $F : \mathbf{X} \to \mathbf{Set}$ is called representable whenever $F \simeq \mathbf{X}(X,-)$.

Examples

- For $F\colon \mathbf{Ord} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(1,-)$.
- For $F\colon \mathbf{Vec} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(\mathbb{R},-)$.

Some facts

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y.$

Definition

A functor $F : \mathbf{X} \to \mathbf{Set}$ is called representable whenever $F \simeq \mathbf{X}(X,-)$.

Examples

- For $F\colon \mathbf{Ord} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(1,-)$.
- For $F\colon \mathbf{Vec} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(\mathbb{R},-)$.

Some facts

- Representable functors preserve limits.

Representable functors

Example

For every category $\mathbf{X}$, $\mathbf{X}(-,-)\colon \mathbf{X}^{\mathrm{op}} \times \mathbf{X} \to \mathbf{Set}$ is a functor.

Remark

$\mathbf{X}(X,-) \simeq \mathbf{X}(Y,-) \implies X \simeq Y$.

Definition

A functor $F : \mathbf{X} \to \mathbf{Set}$ is called representable whenever $F \simeq \mathbf{X}(X,-)$.

Examples

- For $F\colon \mathbf{Ord} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(1,-)$.
- For $F\colon \mathbf{Vec} \to \mathbf{Set}$: $F \simeq \mathbf{Vec}(\mathbb{R},-)$.

Some facts

- Representable functors preserve limits.
- Representable functors preserve monos.

Adjunction

The slogan is “Adjoint functors arise everywhere”.[a]

[a] Saunders MacLane (1998). *Categories for the working mathematician*. 2nd ed. New York: Springer-Verlag, pp. ix + 262.

Adjunction

The slogan is "Adjoint functors arise everywhere".[a]

[a] Saunders MacLane (1998). *Categories for the working mathematician*. 2nd ed. New York: Springer-Verlag, pp. ix + 262.

Definition

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. Then F is left adjoint to G, or G is right adjoint to F, written as $F \dashv G$, whenever

$$\mathbf{B}(FA, B) \simeq \mathbf{A}(A, GB), \qquad (f \longmapsto \overline{f})$$

naturally in A and B. An adjunction between F and G is a choice of such a natural isomorphism.

Adjunction

The slogan is "Adjoint functors arise everywhere".[a]

[a] Saunders MacLane (1998). *Categories for the working mathematician*. 2nd ed. New York: Springer-Verlag, pp. ix + 262.

Definition

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. Then F is left adjoint to G, or G is right adjoint to F, written as $F \dashv G$, whenever

$$\mathbf{B}(FA, B) \simeq \mathbf{A}(A, GB), \qquad (f \longmapsto \overline{f})$$

naturally in A and B. An adjunction between F and G is a choice of such a natural isomorphism.

Remark

The naturality of $f \mapsto \overline{f}$ can be expressed as:

Adjunction

The slogan is "Adjoint functors arise everywhere".[a]

[a] Saunders MacLane (1998). *Categories for the working mathematician*. 2nd ed. New York: Springer-Verlag, pp. ix + 262.

Definition

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. Then F is left adjoint to G, or G is right adjoint to F, written as $F \dashv G$, whenever

$$\mathbf{B}(FA, B) \simeq \mathbf{A}(A, GB), \qquad (f \longmapsto \overline{f})$$

naturally in A and B. An adjunction between F and G is a choice of such a natural isomorphism.

Remark

The naturality of $f \mapsto \overline{f}$ can be expressed as:

1. For every $h\colon A' \to A$: $\overline{f \cdot Fh} = \overline{f} \cdot h$.

Adjunction

The slogan is “Adjoint functors arise everywhere”.[a]

[a] Saunders MacLane (1998). *Categories for the working mathematician*. 2nd ed. New York: Springer-Verlag, pp. ix + 262.

Definition

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. Then F is left adjoint to G, or G is right adjoint to F, written as $F \dashv G$, whenever

$$\mathbf{B}(FA, B) \simeq \mathbf{A}(A, GB), \qquad (f \longmapsto \overline{f})$$

naturally in A and B. An adjunction between F and G is a choice of such a natural isomorphism.

Remark

The naturality of $f \mapsto \overline{f}$ can be expressed as:

1. For every $h\colon A' \to A$: $\overline{f \cdot Fh} = \overline{f} \cdot h$.
2. For every $k\colon B \to B'$: $\overline{k \cdot f} = Gk \cdot \overline{f}$.

Adjunction via (co)units

Theorem

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. There is a bijective correspondence between

1. *Adjunctions $F \dashv G$.*
2. *Natural transformations $\eta\colon 1 \to GF$ and $\varepsilon\colon FG \to 1$ so that*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \varepsilon_F} \\ & & F \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & \xrightarrow{\varepsilon_G} & GFG \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array}$$

Adjunction via (co)units

Theorem

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. There is a bijective correspondence between

1. *Adjunctions $F \dashv G$.*
2. *Natural transformations $\eta\colon 1 \to GF$ and $\varepsilon\colon FG \to 1$ so that*

$$
\begin{array}{ccc}
F & \xrightarrow{F\eta} & FGF \\
 & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \varepsilon_F} \\
 & & F
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccc}
G & \xrightarrow{\varepsilon_G} & GFG \\
 & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle G\varepsilon} \\
 & & G
\end{array}
$$

Corollary

$F \dashv G$ and $F' \dashv G$ implies $F \simeq F'$.

Adjunction via (co)units

Theorem

Let $G\colon \mathbf{B} \to \mathbf{A}$ and $F\colon \mathbf{B} \to \mathbf{A}$ be functors. There is a bijective correspondence between

1. *Adjunctions $F \dashv G$.*
2. *Natural transformations $\eta\colon 1 \to GF$ and $\varepsilon\colon FG \to 1$ so that*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \varepsilon_F} \\ & & F \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & \xrightarrow{\varepsilon_G} & GFG \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array}$$

Corollary

$F \dashv G$ and $F' \dashv G$ implies $F \simeq F'$.

Remark

A category $\mathbf{C}$ has limits of type I if and only if $\Delta\colon \mathbf{C} \to \mathbf{C}^I$ has a right adjoint.

Adjunctions via initial objects

Theorem

A functor $G\colon \mathbf{B} \to \mathbf{A}$ is right adjoint if and only if the category $(A \Rightarrow G)$ has an initial object.

Theorem (General Adjoint Functor Theorem)

Let $G\colon \mathbf{B} \to \mathbf{A}$ be a functor so that $(A \Rightarrow G)$ has a weak initial set. Then

G is right adjoint if and only if G preserves limits.

Adjunctions via initial objects

Theorem

A functor $G\colon \mathbf{B} \to \mathbf{A}$ *is right adjoint if and only if the category* $(A \Rightarrow G)$ *has an initial object.*

Theorem (General Adjoint Functor Theorem)

Let $G\colon \mathbf{B} \to \mathbf{A}$ *be a functor so that* $(A \Rightarrow G)$ *has a weak initial set. Then*

G is right adjoint if and only if G preserves limits.

The proof is based on the following lemmas.

Adjunctions via initial objects

Theorem

A functor $G\colon \mathbf{B} \to \mathbf{A}$ is right adjoint if and only if the category $(A \Rightarrow G)$ has an initial object.

Theorem (General Adjoint Functor Theorem)

Let $G\colon \mathbf{B} \to \mathbf{A}$ be a functor so that $(A \Rightarrow G)$ has a weak initial set. Then

G is right adjoint if and only if G preserves limits.

The proof is based on the following lemmas.

Lemma

Let $\mathbf{C}$ be a complete category with a weak initial set $\mathcal{S}$. Then $\mathbf{C}$ has an initial object.

Adjunctions via initial objects

Theorem

A functor $G\colon \mathbf{B} \to \mathbf{A}$ is right adjoint if and only if the category $(A \Rightarrow G)$ has an initial object.

Theorem (General Adjoint Functor Theorem)

Let $G\colon \mathbf{B} \to \mathbf{A}$ be a functor so that $(A \Rightarrow G)$ has a weak initial set. Then

$$G \text{ is right adjoint if and only if } G \text{ preserves limits.}$$

The proof is based on the following lemmas.

Lemma

Let $\mathbf{C}$ be a complete category with a weak initial set $\mathcal{S}$. Then $\mathbf{C}$ has an initial object.

Lemma

Let $G\colon \mathbf{B} \to \mathbf{A}$ be a limit-preserving functor and assume that $\mathbf{B}$ is complete. Then $(A \Rightarrow G)$ is complete.